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Integrable mappings of the plane preserving biquadratic invariant curves

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Abstract

We provide a general framework to construct integrable mappings of the plane that preserve a one-parameter family $B(x, y, K)$ of biquadratic invariant curves where parametrization by K is very general. These mappings are reversible by construction (i.e. they are the composition of two involutions) and can be shown to be measure preserving. They generalize integrable maps previously given by McMillan and Quispel, Roberts and Thompson. By considering a transformation of the case of the symmetric biquadratic to a canonical form, we provide a normal form for the symmetric integrable map acting on each invariant curve. We give a Lax pair for a large subclass of our symmetric integrable maps, including at least a 10-parameter subfamily of the 12-parameter symmetric Quispel–Roberts–Thompson maps.

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1. Introduction

In recent years there has been growing interest in the study of time-discrete integrable systems (integrable mappings or maps). In the first instance, this can be explained by the numerous areas of physics in which integrable mappings prominently feature. These include: (i) integrable transformations that arise from exactly solvable lattice models in statistical mechanics [1]; (ii) discrete analogues of integrable systems in classical mechanics or solid state physics including the top, elliptic billiards and discrete Heisenberg spin chains [9, 16, 19, 20]; (iii) integrable mappings obtained from reductions of differential-difference soliton equations [14, 15] or discretizations of integrable partial differential equations [12, 13].

As well as being of physical interest, the study of integrable systems has intrinsic mathematical appeal. Some of the topics that have been investigated for such systems include

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(cf the collection [4] and the excellent review [5] and references therein): necessary and sufficient conditions for integrability, discrete Painleve equations, Lax pairs, discrete analogues of Hamiltonian structures and quantization. In all these areas of investigation, it has proved useful to have sample classes of integrable maps with which to work.

The present paper aims to extend a large class of planar (i.e. two-dimensional) integrable maps found previously and, in the process, to show how both the previously found and new integrable maps can be viewed in a common framework. We call a planar mapping integrable if it is measure preserving and leaves invariant each curve of a one-parameter family of non-intersecting curves

$$C(x, y, K) = 0 \tag{1}$$

that foliate the plane. Each invariant curve of the family is parametrized by K and is left invariant so that $K = k(x, y)$ defined explicitly or implicitly by solving (1) for K is an integral for the map.

Consider the following family of biquadratic curves in the plane:

$$\begin{aligned}(\alpha_0 + \alpha_1 K)x^2y^2 + (\beta_0 + \beta_1 K)x^2y + (\delta_0 + \delta_1 K)xy^2 + (\gamma_0 + \gamma_1 K)x^2 + (\kappa_0 + \kappa_1 K)y^2 \\ +(\epsilon_0 + \epsilon_1 K)xy + (\xi_0 + \xi_1 K)x + (\lambda_0 + \lambda_1 K)y + (\mu_0 + \mu_1 K) = 0.\end{aligned} \tag{2}$$

For any choice of the 18 arbitrary parameters $\alpha_i, \ldots, \mu_i (i = 0, 1)$, the family of curves (2) gives a foliation of the x–y plane. This follows since (2) is linear in K and so uniquely determines K for each point (x, y) (with the possible exception of a single curve of points; see the end of section 2). In [14, 15] integrable rational mappings of the plane (called the QRT family in [5]) were given which preserve the foliation (2). These maps take the form

$$x' = \frac{f_1(y) - xf_2(y)}{f_2(y) - xf_3(y)} \qquad y' = \frac{g_1(x') - yg_2(x')}{g_2(x') - yg_3(x')} \tag{3}$$

where the functions f_i and g_i $(i = 1, 2, 3)$ are certain quartic polynomials whose coefficients are functions of the 18 parameters (given explicitly in (34)).

When $\delta_i = \beta_i, \kappa_i = \gamma_i, \lambda_i = \xi_i$ in (2), each curve of the foliation becomes symmetric in x and y. In this case, $g_i = f_i$ in (3) and the mapping (3) corresponds to two applications of the integrable symmetric QRT mapping

$$x' = y \qquad y' = \frac{f_1(y) - xf_2(y)}{f_2(y) - xf_3(y)}. \tag{4}$$

The QRT mappings (3) and (4) are measure preserving as well as being reversible (i.e. the composition of two involutions) [14–16]. The symmetric map (4) is a generalization of a reversible area-preserving integrable map constructed by McMillan [11]. McMillan's family of integrable maps preserve the foliation (2) when it is symmetric in x and y and $\alpha_1, \beta_1, \ldots, \lambda_1$ are all zero. Consequently K only appears in the form $\mu_0 + \mu_1 K$ which can be reparametrized to the case $\mu_0 = 0, \mu_1 = 1$ by $K \to \mu_0 + \mu_1 K$.

The purpose of this paper is to present integrable mappings of the plane that preserve quite general biquadratic foliations of the plane of the form (2) with the linear functions of K replaced by very general functions of K. In principle, these families can depend on an arbitrarily large number of parameters (as distinct from the 18 parameters in the QRT family). These new integrable mappings are reversible and measure preserving and contain the QRT mappings (3) and (4) as special cases.

It is very significant that the integrable mappings presented here can be formulated as generalized versions of the form of mapping originally considered by McMillan or of an asymmetric counterpart given in [18, appendix A]. To be more explicit, on each curve of the invariant biquadratic foliation, the mapping acts as (what we will call) a symmetric or asymmetric McMillan map. However, this McMillan map changes from curve to curve so we call the new integrable maps *curve-dependent McMillan* (CDM) maps. In particular, the QRT maps can be seen as particular cases of them. Although the global explicit form of the CDM mappings presented here is often not written down, this proves irrelevant to many features of the dynamics.

The plan of the paper is as follows. In sections 2 and 3, we define the maps and give examples of them, including showing how the QRT maps are a special case. In section 4, we discuss the issue of a normal form for the dynamics when the biquadratic foliation is symmetric in x and y. In section 5, we give a Lax pair for a large subclass of maps of the form (4). More details of the maps presented here, particularly the ones that preserve an asymmetric foliation, will be given in [8].

2. Curve-dependent McMillan form and preservation of biquadratic curves

Consider the function

$$\begin{aligned} B(x, y, K) = {} & \alpha(K)x^2y^2 + \beta(K)x^2y + \delta(K)xy^2 + \gamma(K)x^2 + \kappa(K)y^2 \\ & + \epsilon(K)xy + \xi(K)x + \lambda(K)y + \mu(K) \end{aligned} \tag{5}$$

where the nine coefficients $\alpha, \ldots, \mu$ are in general functions of a parameter K. For each K

$$B(x, y, K) = 0 \tag{6}$$

defines a biquadratic curve in the (x, y) plane. We can alternatively write (6) using a dot product

$$B(x, y, K) = X \cdot A(K)\,Y = 0 \tag{7}$$

where

$$X := \begin{pmatrix} x^2 \\ x \\ 1 \end{pmatrix} \qquad Y := \begin{pmatrix} y^2 \\ y \\ 1 \end{pmatrix} \tag{8}$$

and

$$A(K) := \begin{pmatrix} \alpha(K) & \beta(K) & \gamma(K) \\ \delta(K) & \epsilon(K) & \xi(K) \\ \kappa(K) & \lambda(K) & \mu(K) \end{pmatrix}. \tag{9}$$

The following proposition establishes, for any value of the parameter K, two automorphisms of the curve (6).

Proposition 1. *Let $(x', y') = G_i(K)\,(x, y)$ be the image of (x, y) under either of the following involutions, parametrized by K:*

$$G_1(K): \quad x' = x \qquad y' = -y - \frac{\beta(K)x^2 + \epsilon(K)x + \lambda(K)}{\alpha(K)x^2 + \delta(K)x + \kappa(K)} \tag{10}$$

and

$$G_2(K): \quad x' = -x - \frac{\delta(K)y^2 + \epsilon(K)y + \xi(K)}{\alpha(K)y^2 + \beta(K)y + \gamma(K)} \qquad y' = y. \tag{11}$$

Both involutions satisfy

$$B(x', y', K) = B(x, y, K) \tag{12}$$

with $B(x, y, K)$ given by (5). In particular, if (x, y) satisfy $B(x, y, K) = 0$ for some fixed K, then $B(x', y', K) = 0$. G_1 and G_2 are the most general non-identity automorphisms fixing one coordinate that satisfy (12).

Proof. Consider $G_1(K)$. It is easy to check that it is an involution for any value of K. Furthermore, from (10) we have (suppressing the K dependence for convenience)

$$(y' + y)(\alpha x^2 + \delta x + \kappa) + (\beta x^2 + \epsilon x + \lambda) = 0. \tag{13}$$

Multiplying both sides of (13) by $(y' - y)$ and separating terms involving y' and y gives

$$y'^2(\alpha x^2 + \delta x + \kappa) + y'(\beta x^2 + \epsilon x + \lambda) = y^2(\alpha x^2 + \delta x + \kappa) + y(\beta x^2 + \epsilon x + \lambda). \tag{14}$$

The latter is equivalent to (12) if $x' = x$. Conversely, working backwards from (12), it is also clear that $G_1(K)$ is the unique non-identity automorphism that both fixes the x coordinate and preserves $B(x, y, K)$. A particular case of (12) is that $G_1(K)$ is an automorphism of the curve $B(x, y, K) = 0$.

Similar reasoning establishes the analogous result for G_2. □

Since $G_1(K)$ and $G_2(K)$ preserve the biquadratic curve (7) for each K, then so does their composition $M_{\mathrm{a}}(K) := G_1(K) \circ G_2(K)$ given by

$$\begin{aligned} M_{\mathrm{a}}(K): \quad x' &= -x - \frac{\delta(K)y^2 + \epsilon(K)y + \xi(K)}{\alpha(K)y^2 + \beta(K)y + \gamma(K)} \\ y' &= -y - \frac{\beta(K)x'^2 + \epsilon(K)x' + \lambda(K)}{\alpha(K)x'^2 + \delta(K)x' + \kappa(K)}. \end{aligned} \tag{15}$$

We use the subscript 'a' in $M_{\mathrm{a}}(K)$ to refer to the fact that curve (6) or (7) is *asymmetric* in x and y (equivalently, the matrix A in (9) is asymmetric). When the biquadratic (6) is symmetric in x and y, we can write it with six coefficients

$$\begin{aligned} B_{\mathrm{s}}(x, y, K) = \alpha(K)x^2y^2 + \beta(K)(x^2y + xy^2) + \gamma(K)(x^2 + y^2) \\ + \epsilon(K)xy + \xi(K)(x + y) + \mu(K) = 0 \end{aligned} \tag{16}$$

equivalently the matrix of (9) is now symmetric: $A(K) = A(K)^{\mathrm{T}}$ (superscript T denotes matrix transpose) with $\delta(K) = \beta(K)$, $\kappa(K) = \gamma(K)$ and $\lambda(K) = \xi(K)$. In this case, another involutory automorphism of the curve $B_{\mathrm{s}}(x, y, K) = 0$ is clearly

$$G_{\mathrm{s}}: \quad x' = y \qquad y' = x. \tag{17}$$

It follows that the mapping $M_{\mathrm{s}}(K) := G_1(K) \circ G_{\mathrm{s}}$ given by

$$M_{\mathrm{s}}(K): \quad x' = y \qquad y' = -x - \frac{\beta(K)y^2 + \epsilon(K)y + \xi(K)}{\alpha(K)y^2 + \beta(K)y + \gamma(K)} \tag{18}$$

is also an automorphism of the curve (and in fact in this case (15) corresponds to $M_{\mathrm{s}}(K)^2$).

Consider varying K in (7)–(9) to give a one-parameter *family* of curves in the plane. The maps $M_{\mathrm{a}}(K)$ or $M_{\mathrm{s}}(K)$ then represent a one-parameter family of curve automorphisms, each preserving the corresponding biquadratic curve in the family.

An important motivation for us is the previously discovered special case of the family of curves obtained from taking (6) or (7)–(9) when $\alpha(K) = \alpha_0$, $\beta(K) = \beta_0, \ldots, \epsilon(K) = \epsilon_0$, $\lambda(K) = \lambda_0$ are constants (i.e. independent of K) and $\mu(K) = K$ as follows:

$$\alpha_0 x^2y^2 + \beta_0 x^2 y + \delta_0 xy^2 + \gamma_0 x^2 + \kappa_0 y^2 + \epsilon_0 xy + \xi_0 x + \lambda_0 y + K = 0. \tag{19}$$

The family (19) can be used to provide a *foliation* of the plane, i.e. each point (x, y) belongs on a unique biquadratic curve specified by its value K. The range of K needed to include all points of the plane can be readily calculated: for example, if α_0, γ_0, κ_0 are all positive, $K \in (-\infty, K_{\max})$, $K_{\max}$ constant, whereas if α_0, γ_0, κ_0 are non-zero and of mixed sign, $K \in (-\infty, +\infty)$. Since $\mu(K)$ does not appear in (15) and (18), it follows in the case of (19) that $M_a(K)$ ($M_s(K)$) are in fact K*-independent*, taking the same form on every curve of the foliation. Consequently, we can define *global* rational mappings of the plane

$$M_a: \quad x' = -x - \frac{\delta_0 y^2 + \epsilon_0 y + \xi_0}{\alpha_0 y^2 + \beta_0 y + \gamma_0}$$
$$y' = -y - \frac{\beta_0 x'^2 + \epsilon_0 x' + \lambda_0}{\alpha_0 x'^2 + \delta_0 x' + \kappa_0} \tag{20}$$

respectively,

$$M_s: \quad x' = y \qquad y' = -x - \frac{\beta_0 y^2 + \epsilon_0 y + \xi_0}{\alpha_0 y^2 + \beta_0 y + \gamma_0} \tag{21}$$

that preserve the biquadratic foliation (19) in general, respectively, when (19) is symmetric in x and y. It is easy to check that M_a or M_s are area preserving. Furthermore, it is clear by construction that M_a or M_s are reversible mappings since they are compositions of two involutions: $M_a = G_1 \circ G_2$, $M_s = G_1 \circ G_s$ where G_1 and G_2 are the obvious anti area-preserving maps of the plane obtained from the K-independent versions of $G_1(K)$ and $G_2(K)$ extended to the whole plane (cf [10, 18] for properties of reversible dynamical systems).

The map M_s of (21) was discovered by McMillan [11] and we will call it the *symmetric McMillan map*[1]. The map M_a of (20) was given in [18, appendix A] and we will call it the *asymmetric McMillan map*. Note that the symmetric McMillan map M_s and asymmetric McMillan map M_a are just area-preserving cases of the symmetric QRT map (4) and asymmetric QRT map (3), respectively.

What we now have in mind is to consider the general family of curves (6) or (7)–(9) where *all* the coefficients are possibly functions of K. Typically curves from this family will intersect each other with some points of the plane belonging to multiple curves and other points not contained on any curves of the family. However, we will assume that the following condition is satisfied so as to *guarantee* that the family defines a foliation of the plane.

Condition F. The equation $B(x, y, K) = 0$ defines globally $K = k(x, y)$ as a smooth real-valued function for all (x, y).

It is certainly not necessary that one can actually solve globally for $k(x, y)$, rather we want a global implicit function theorem that just says that such a function exists. In the following section, we will show that imposing

$$\frac{\partial B}{\partial K}(x, y, K) = X \cdot \frac{\mathrm{d}A(K)}{\mathrm{d}K} \quad Y > 0 \tag{22}$$

for all (x, y, K) is a sufficient and easy-to-apply condition that gives condition F when $\alpha(K), \ldots, \mu(K)$ are polynomial or rational functions of K (with care (22) can also be used to help construct examples when some of the coefficient functions are not rational). Furthermore, we show that condition F can actually be relaxed somewhat.

Assume then that condition F is satisfied so that a foliation of the plane by biquadratic curves exists. With condition F satisfied, $M_a(K)$ and $M_s(K)$ can be used to create smooth global maps of the plane that preserve each curve of the foliation by biquadratic curves (6), respectively (16). Specifically, these new maps of the plane which we denote L_a, respectively L_s, are formed by solving $B(x, y, K) = 0$ for $K = k(x, y)$ and substituting in $M_a(K)$,

[1] McMillan [11] actually discovered the integrable mapping M_s with $\xi_0 \equiv 0$.

respectively $M_s(K)$, so that

$$\begin{aligned} L_a := M_a(k(x,y)) : \quad x' &= -x - \left.\frac{\delta(K)y^2 + \epsilon(K)y + \xi(K)}{\alpha(K)y^2 + \beta(K)y + \gamma(K)}\right|_{K=k(x,y)} \\ y' &= -y - \left.\frac{\beta(K)x'^2 + \epsilon(K)x' + \lambda(K)}{\alpha(K)x'^2 + \delta(K)x' + \kappa(K)}\right|_{K=k(x,y)} \end{aligned} \tag{23}$$

and

$$\begin{aligned} L_s := M_s(k(x,y)) : \quad x' &= y \\ y' &= -x - \left.\frac{\beta(K)y^2 + \epsilon(K)y + \xi(K)}{\alpha(K)y^2 + \beta(K)y + \gamma(K)}\right|_{K=k(x,y)}. \end{aligned} \tag{24}$$

We will use, for brevity, the notation $|_{K=k(x,y)}$ to indicate that K is replaced by $k(x, y)$. Sometimes it will be possible to solve explicitly for $k(x, y)$ and obtain closed form expressions for L_a and L_s. However, L_a and L_s are still well defined maps of the plane for $k(x, y)$ defined implicitly. For example, the numerical calculation of the orbit of (x_0, y_0) under L_a involves solving $B(x_0, y_0, K) = 0$ to find $K = k_0 = k_0(x_0, y_0)$ and then iterating $M_a(k_0)$ for the initial point. Examples of both implicit and explicit calculation of $k(x, y)$ will be given below. In either case, since L_a and L_s both satisfy $B(x, y, K) = 0 \Rightarrow B(x', y', K) = 0$, they possess the integral

$$k(x', y') = k(x, y). \tag{25}$$

It is clear from their definitions that L_a and L_s are in general certainly *not* rational mappings over the plane (e.g. see example 2 of the next section). However, significantly, for each curve of the foliation (i.e. for each fixed K) the action of L_a or L_s is rational on the curve, indeed the dynamics of L_a (L_s) on each curve is that of an asymmetric (symmetric) McMillan map. For this reason, when condition F holds and the mapping equations in (23), (24) are dependent on K, we will call L_a (L_s) an asymmetric (symmetric) curve-dependent McMillan map (CDM). Of course, in this general situation, L_a and L_s are, by construction, also reversible mappings since they are the composition of two involutions: $L_a = G_1(k(x, y)) \circ G_2(k(x, y))$ and $L_s = G_1(k(x, y)) \circ G_s$.

Another way to view the dynamics of the CDMs L_a or L_s is in the three-dimensional space (x, y, K) with the mapping equations of (15) or (18) augmented by adding the dynamics $K' = K$ in the third coordinate. These associated three-dimensional maps preserve, from (12), the surfaces $B(x, y, K) = C$ with C an arbitrary constant, which foliate the three-dimensional space. Condition F means that the surface $B(x, y, K) = 0$ is the graph of the function $k(x, y)$ and the CDMs are the restriction of the associated three-dimensional maps to the invariant surface $B(x, y, K) = 0$ followed by projection onto the (x, y) plane. On each plane with K constant, $M_a(K)$ ($M_s(K)$) can be considered not only as an automorphism of the curve $B(x, y, K) = 0$ but extends to a global rational mapping of the plane equal to the asymmetric (symmetric) McMillan map that preserves the entire biquadratic foliation of the plane $B(x, y, K) = C$ ($B_s(x, y, K) = C$). The foliation (7) that we are interested in corresponds to choosing one curve (that with $C = 0$) from each such foliation and projecting it down to the (x, y) plane; with condition F satisfied, this set of curves is a new foliation of the plane since it is precisely the set of level curves of $k(x, y)$.

This geometric viewpoint in the (x, y, K) space can be used to show that the CDM mappings L_a or L_s are measure preserving. Recall that $L : (x, y) \mapsto (x', y')$ is measure preserving [18] with density $m(x, y)$ if the Jacobian determinant $J(x, y) := \det \mathrm{d}L(x, y)$ can be written

$$J(x, y) = \pm \frac{m(x, y)}{m(x', y')} \Rightarrow \int_A m(x, y)\, \mathrm{d}x\, \mathrm{d}y = \pm \int_{L(A)} m(x', y')\, \mathrm{d}x' \mathrm{d}y' \tag{26}$$

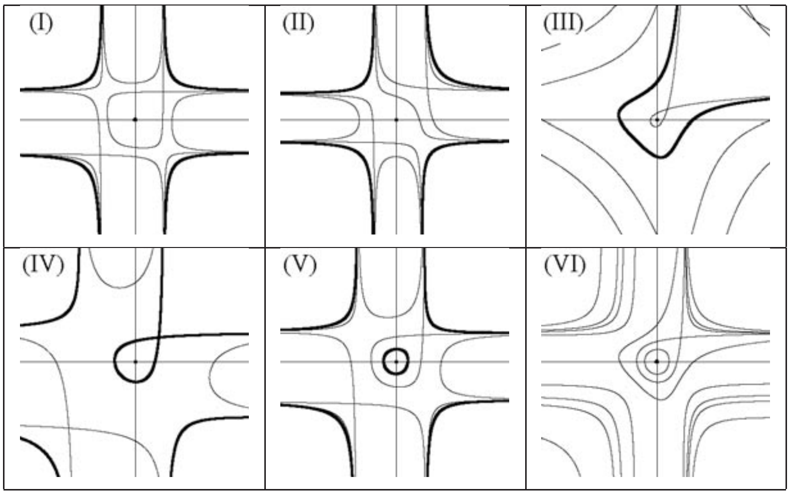

Figure 1. The phase portrait of the symmetric CDM map L_1 of example 1 on the square $[-5, 5] \times [-5, 5]$ is shown in (VI) in the bottom right-hand corner. Pictures (I)–(V) show the phase portraits of the five McMillan maps obtained from $M_1(K)$ by respectively fixing $K = K_0$ with $K_0 \in \{3, 0.411\,76, -0.206\,57, -0.829\,82, -4.561\,52\}$. The bold curve in each of (I)–(V) is $B_1(x, y, K_0) = 0$, and the sequence shows how the phase portrait of L_1 is built up from one curve taken from the phase portrait of each McMillan map.

for any region A in the plane (the $\pm$ sign is taken according to whether L is orientation preserving/reversing; area preservation corresponds to $m(x, y) \equiv 1$). In [8] it is shown that L_a and L_s satisfy (26) with

$$m(x, y) = \left[\left. \frac{\partial B}{\partial K}(x, y, K) \right|_{K=k(x,y)} \right]^{-1} = \left[X \cdot \frac{\mathrm{d}A}{\mathrm{d}K}(k(x, y))\ Y \right]^{-1}. \qquad (27)$$

To illustrate the above discussion, consider the following explicit example of a CDM.

Example 1.

$$B_1(x, y, K) = (3K + 1)x^2y^2 + (K - 1)(x^2y + xy^2) - 6K(x^2 + y^2) - xy + K - 3. \qquad (28)$$

Since $B_1(x, y, K)$ is symmetric in x and y, the CDM map L_1 that preserves the foliation $B_1(x, y, K) = 0$ takes the L_s form. Denoting the solution of $B_1(x, y, K) = 0$ by $K = k_1(x, y)$, we have

$$\begin{aligned} L_1 = M_1(K)\,|_{K=k_1(x,y)} :\quad & x' = y \\ & y' = -x - \left. \frac{(K - 1)y^2 - y}{(3K + 1)y^2 + (K - 1)y - 6K} \right|_{K=k_1(x,y)}. \end{aligned} \qquad (29)$$

Figure 1 shows how the phase portrait of the CDM map L_1 is built up from one curve taken from the phase portrait of each McMillan map $M_1(K)$. In this example, it is clearly possible to solve $B_1(x, y, K) = 0$ explicitly for $k_1(x, y)$:

$$k_1(x, y) = -\frac{x^2y^2 - (x^2y + xy^2) - xy - 3}{3x^2y^2 + (x^2y + xy^2) - 6(x^2 + y^2) + 1}. \qquad (30)$$

Substituting this into $M_1(K)$ gives an alternative explicit form for the map L_1:

$$\begin{aligned} L_1 = M_1(k(x, y)) :\quad & x' = y \\ & y' = \frac{y(6y^3 + 6y^2 + 2y - 1) - 2x(3y^4 - 3y^3 - 5y^2 - y + 9)}{2(3y^4 - 3y^3 - 5y^2 - y + 9) - xy(4y^3 + 3y^2 - 5y - 6)}. \end{aligned} \qquad (31)$$

The explicit form (31) of the CDM L_1 shows that it is a QRT map (4). This is true in general: the QRT family [14–16] can be rederived as a particular case of CDM maps. The foliation (2) is a special case of (6) with $\alpha(K) = \alpha_0 + K\alpha_1, \ldots, \mu(K) = \mu_0 + K\mu_1$ being affine functions of K with $\alpha_i, \ldots, \mu_i$ $(i = 0, 1)$ being 18 arbitrary constants. The foliation (2) can be written

$$B_{\mathrm{QRT}}(x, y, K) = X \cdot (A_0 + KA_1)Y = X \cdot A_0 Y + K(X \cdot A_1 Y) = 0 \tag{32}$$

where A_0 and A_1 are constant coefficient matrices of the form (9). Provided $X \cdot A_1 Y \neq 0$, we can solve (32) explicitly for K to find

$$k(x, y) = -\frac{X \cdot A_0 Y}{X \cdot A_1 Y}. \tag{33}$$

If we substitute this expression for $k(x, y)$ into (23) with $\alpha(K) = \alpha_0 + K\alpha_1, \ldots, \lambda(K) = \lambda_0 + K\lambda_1$, we obtain after manipulation the asymmetric QRT mapping form (3) where f_i and g_i can be neatly expressed as components of cross products

$$(f_1, f_2, f_3)(y) = (A_0 Y) \times (A_1 Y) \qquad (g_1, g_2, g_3)(x') = (A_0^{\mathrm{T}} X') \times (A_1^{\mathrm{T}} X'). \tag{34}$$

Similarly, substituting (33) into L_{s} of (24) when A_0 and A_1 are symmetric results in (4) with f_i of (34). It is seen that Example 1 is precisely an example of a symmetric CDM L_{s} which can also be written in the QRT form (31). Measure preservation of the QRT maps (3) and (4) with $m(x, y) = [X \cdot A_1 Y]^{-1}$ (which is consistent with (27)) was previously proved in [16, 18].

Whenever (x, y) satisfies $X \cdot A_1 Y = 0$ in (32), $B_{\mathrm{QRT}}(x, y, K) = X \cdot A_0 Y$ is no longer a function of K (consistent with (22) failing since, for the QRT foliation (32), $\frac{\partial B}{\partial K}(x, y, K) = X \cdot A_1 Y$). The set of points on the curve $X \cdot A_1 Y = 0$ satisfying $X \cdot A_0 Y \neq 0$ does not strictly belong to the foliation (32) but can be considered to have K infinite from (33). This set of points, if non-empty, is necessarily mapped to itself by the corresponding QRT map. The set of (at most eight) points satisfying $X \cdot A_1 Y = X \cdot A_0 Y = 0$ satisfy (32) for *any* value of K and, in the phase portrait of the corresponding QRT map, it is found that many curves of the foliation pass through these points [15, figure 2]. In this way it is seen that the QRT maps are examples in which the condition F and the associated (22) are relaxed somewhat from stated above.

Finally, to end this section, we make a couple of technical remarks.

Remark 1. This concerns the generality of defining CDMs in terms of preserving the foliation (7). If (22) is satisfied (and the extra assumptions mentioned in the next section apply), then $B(x, y, K) = C$, with C a fixed constant, also defines globally $K = k_C(x, y)$ as a function of x and y (so the $k(x, y)$ in the discussion above is $k_0(x, y)$) and this leads to another foliation of the x–y plane which from (12) is preserved by (23) with $k \to k_C$. In fact, this can be further extended and the CDMs can be viewed as the general solution to the following problem: from a one-parameter family of McMillan maps $M_{\mathrm{a}}(K)$ in (x, y, K) space, is it possible to create a new biquadratic foliation of the plane by choosing one curve $B(x, y, K) = C(K)$, where $C(K)$ is a smooth function of K, from the phase portrait of each McMillan map in the family (for some range of K)? This is possible if and only if the same one-parameter family allows a biquadratic foliation $B^{\dagger}(x, y, K) = 0$ where $B^{\dagger}(x, y, K) = B(x, y, K) - C(K)$ differs from $B(x, y, K)$ only in the coefficient $\mu^{\dagger}(K) = \mu(K) - C(K)$. From the fact that $M_{\mathrm{a}}(K)$ and $M_{\mathrm{s}}(K)$ do not depend on $\mu(K)$, (12) implies $B^{\dagger}(x', y', K) = B^{\dagger}(x, y, K)$ for these maps and vice versa.

Remark 2. As the biquadratic (2) or (32) for the QRT family illustrates, the biquadratics $B(x, y, K) = 0$ we are considering may depend on additional parameters other than K which enter into their coefficient functions $\alpha(K), \ldots, \mu(K)$ etc (see also (42) and example 2 below).

We choose to suppress this and just indicate the distinguished parameter K which is solved from $B(x, y, K) = 0$ (whence $K = k(x, y)$ will in general depend on the additional parameters).

3. Condition for foliation and examples

In this section, we discuss how (22) is sufficient to guarantee condition F and give some more examples of the integrable CDM mappings L_a and L_s.

Consider a fixed x and y. Then $B(x, y, K)$ of (5) is a function of K built up as a linear combination of the coefficient functions $\alpha(K), \ldots, \mu(K)$. We assume that the coefficient functions are defined and smooth for all K so that $B(x, y, K)$ is also smooth and that its graph is connected. When (22) holds for all (x, y, K), it implies that $B(x, y, K)$ is a monotone increasing function of K and hence injective[2]. It follows that if there exists a solution to (6) so that $B(x, y, K)$ has a zero and then this value of K is unique.

If the coefficient functions $\alpha(K), \ldots, \mu(K)$ in (5) are polynomial or everywhere-defined rational functions, then injectivity of $B(x, y, K)$ as a function of K implies surjectivity. Hence for every (x, y) in the plane, there is a *unique* solution K to $B(x, y, K) = C$ for C constant. In particular this is true for the case $C = 0$ of (6) so that every point in the plane belongs to a unique curve from the family (6) meaning that they foliate the plane.

Suppose the coefficient functions $\alpha(K), \ldots, \mu(K)$ are not all polynomial or everywhere-defined rational functions of K. Since in general an everywhere-defined injective real function is not surjective and need not have a zero (for example $\exp(K)$), we need to impose extra conditions on $B(x, y, K)$ in order to guarantee a solution to (6). For example, it is sufficient to check that for any (x, y) it is positive at some value of K and negative at another value or to check the behaviour at infinity ($\lim_{K\to\infty} B(x, y, K) = D_+ > 0$ and $\lim_{K\to-\infty} B(x, y, K) = D_- < 0$).

The above discussion indicates sufficient conditions that guarantee that (6) implies the existence of a global function $K = k(x, y)$ defined everywhere in the plane. Condition (22) also guarantees via the local implicit function theorem that $k(x, y)$ is smooth in every neighbourhood of (x, y). In fact, the knowledge that $B(x, y, K) = 0$ has globally only one solution for each (x, y) guarantees the global smoothness of $k(x, y)$.

We now give some examples of foliations of the plane (6) for which $B(x, y, K)$ satisfies (22) and has, in the case of non-polynomial coefficients, the above-mentioned behaviour to guarantee a zero. Condition (22) requires that the biquadratic $X \cdot \frac{\mathrm{d}A(K)}{\mathrm{d}K} Y$ be positive everywhere. Viewing this as a quadratic polynomial in y say, we can write

$$\frac{\partial B}{\partial K}(x, y, K) = Py^2 + Qy + R \tag{35}$$

with

$$\begin{aligned} P &:= \alpha'(K)x^2 + \delta'(K)x + \kappa'(K) \\ Q &:= \beta'(K)x^2 + \epsilon'(K)x + \lambda'(K) \\ R &:= \gamma'(K)x^2 + \xi'(K)x + \mu'(K). \end{aligned}$$

In (35), we require that $P \neq 0$ implies that the discriminant $\Delta = Q^2 - 4PR$ is negative, or $P = 0$ implies $Q = 0$. Selecting coefficient functions whose derivatives satisfy these conditions can be somewhat complicated in general, but a useful special case is given by

$$B(x, y, K) = x^2y^2 + \gamma(K)x^2 + \kappa(K)y^2 + \epsilon(K)xy + \mu(K). \tag{36}$$

[2] Without loss of generality we avoid the monotone decreasing condition $\frac{\partial B}{\partial K}(x, y, K) < 0$ since it corresponds to $B \to -B$ and the same biquadratic foliation (6).

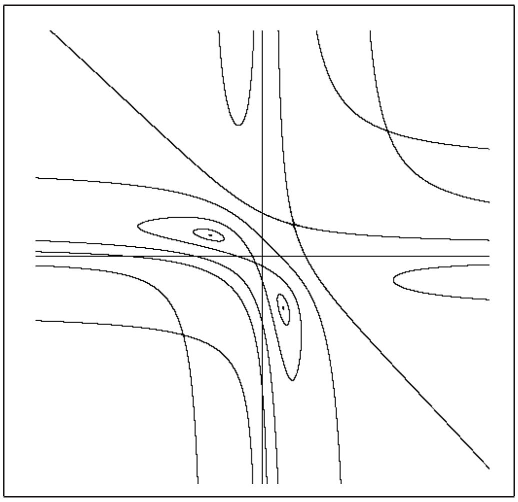

Figure 2. Phase portrait of L_2 of example 2 on the square $[-4, 4] \times [-4, 4]$.

In this case, we have $P = \kappa'(K)$ and

$$\Delta(x, y, K) = (\epsilon'(K)^2 - 4\kappa'(K)\,\gamma'(K))\,x^2 - 4\kappa'(K)\,\mu'(K) \tag{37}$$

so that (22) is satisfied provided

$$\kappa'(K),\ \mu'(K) > 0 \qquad \text{and} \qquad \epsilon'(K)^2 < 4\kappa'(K)\,\gamma'(K). \tag{38}$$

When (36) is symmetric, $\kappa(K) = \gamma(K)$ and (38) becomes

$$\gamma'(K),\ \mu'(K) > 0 \qquad \text{and} \qquad \epsilon'(K)^2 < 4\gamma'(K)^2. \tag{39}$$

A very useful way we have found of creating examples is as follows. Let $r(K)$, $s(K)$, $u(K)$ and $v(K)$ be arbitrary everywhere-defined functions of K and let $w(K)$ be positive for all K. Define

$$\frac{\partial B}{\partial K}(x, y, K) := (r(K)xy + s(K)x + u(K)y + v(K))^2 + w(K) > 0. \tag{40}$$

By construction, (22) is immediately satisfied. The result of expanding (40) and then integrating with respect to K can be written

$$B(x, y, K) = \int_0^K \frac{\partial B}{\partial t}(x, y, t)\,\partial t + X \cdot A_0 Y \tag{41}$$

where A_0 is an arbitrary constant matrix. Equation (41) corresponds to an asymmetric biquadratic $B(x, y, K)$ in general and the symmetric biquadratic $B_s(x, y, K)$ if and only if $s(K) = u(K)$ in (40). We remark that without the addition of $w(K)$, the right-hand side of (40) corresponds to the biquadratic $\frac{\partial B}{\partial K}(x, y, K)$ written in the quadratic form (35) having a repeated zero in y, equivalently having discriminant $\Delta = Q^2 - 4PR$ equal to 0. The effect of $w(K)$ is to push the quadratic $Py^2 + Qy + R$ away from the axis so that it is never zero. The method based on (40) contains essentially five arbitrary functions in the asymmetric case and four in the symmetric case, which compares with the nine coefficient functions of $B(x, y, K)$ of (6) and six of $B_s(x, y, K)$ of (16).

An illustration of (40) and (41) in the symmetric case is to take (40) with $r(K) = \alpha_2 + \alpha_1 K$, $s(K) = u(K) = \beta_2 + \beta_1 K$, $v(K) = \gamma_2 + \gamma_1 K$ and $w(K) = \epsilon_2^2 + \epsilon_1^2 K^2$. Using (41) with

$X \cdot A_0 Y = \alpha_0 x^2 y^2 + \beta_0(x^2 y + x y^2) + \gamma_0(x^2 + y^2) + \epsilon_0 xy + \xi_0(x + y) + \mu_0$, and multiplying through by 3, gives a symmetric biquadratic (16) with

$$\begin{aligned}
\alpha(K) &= \alpha_1^2 K^3 + 3\alpha_1\alpha_2 K^2 + 3\alpha_2^2 K + 3\alpha_0 \\
\beta(K) &= 2\alpha_1\beta_1 K^3 + 3(\alpha_1\beta_2 + \alpha_2\beta_1)K^2 + 6\alpha_2\beta_2 K + 3\beta_0 \\
\gamma(K) &= \beta_1^2 K^3 + 3\beta_1\beta_2 K^2 + 3\beta_2^2 K + 3\gamma_0 \\
\epsilon(K) &= 2(\alpha_1\gamma_1 + \beta_1^2)K^3 + 3(\alpha_1\gamma_2 + \alpha_2\gamma_1 + 2\beta_1\beta_2)K^2 + 6(\alpha_2\gamma_2 + \beta_2^2)K + 3\epsilon_0 \\
\xi(K) &= 2\beta_1\gamma_1 K^3 + 3(\beta_1\gamma_2 + \beta_2\gamma_1)K^2 + 6\beta_2\gamma_2 K + 3\xi_0 \\
\mu(K) &= (\gamma_1^2 + \epsilon_1^2)K^3 + 3\gamma_1\gamma_2 K^2 + 3(\gamma_2^2 + \epsilon_2^2)K + 3\mu_0.
\end{aligned} \tag{42}$$

By construction, this symmetric biquadratic is cubic in K and has 14 additional parameters. The corresponding CDM L_s will have 14 parameters in it. It is clear that by increasing the degree of K in polynomial choices of $r(K)$, $s(K)$, $v(K)$ and $w(K)$ in such a way that the corresponding biquadratic $B(x, y, K)$ has odd degree in K, we can satisfy (22) and create a CDM with arbitrarily many parameters in it.

As with example 1 of the previous section, the examples below correspond to symmetric biquadratic foliations (16) (cf [8] for asymmetric examples). The corresponding integrable CDM maps are given by L_s of (24) after substituting the appropriate coefficient functions $\alpha(K)$, $\beta(K)$, $\gamma(K)$, $\epsilon(K)$ and $\xi(K)$, and finding (implicitly or explicitly) the solution $K = k_i(x, y)$ of $B_s(x, y, K) = 0$.

Example 2. A particular example of (42) is $(\alpha_2, \alpha_1, \alpha_0, \beta_2, \beta_1, \beta_0, \gamma_2, \gamma_1, \gamma_0, \epsilon_2, \epsilon_1, \epsilon_0, \xi_0, \mu_0) = (-6, 0, 7, -3, -1, -6, 2, 4, 4, -5, 1, 9, 2, 1)$, i.e.

$$\begin{aligned}
B_2(x, y, K) = {} & 3(36K + 7)x^2 y^2 + 18(K^2 + 6K - 1)(x^2 y + x y^2) \\
& + (K^3 + 9K^2 + 27K + 12)(x^2 + y^2) + (2K^3 - 54K^2 - 18K + 27)xy \\
& - 2(4K^3 + 21K^2 + 18K - 3)(x + y) + 17K^3 + 24K^2 + 87K + 3 = 0.
\end{aligned} \tag{43}$$

The CDM map L_s preserving this foliation is

$$\begin{aligned}
L_2 :\ & x' = y \\
& y' = -x - \left.\frac{18(K^2 + 6K - 1)y^2 + (2K^3 - 54K^2 - 18K + 27)y - 2(4K^3 + 21K^2 + 18K - 3)}{3(36K + 7)y^2 + 18(K^2 + 6K - 1)y + (K^3 + 9K^2 + 27K + 12)}\right|_{K = k_2(x,y)}.
\end{aligned} \tag{44}$$

Part of the phase portrait of L_2 is shown in figure 2 (the phase portrait is created using [2]). The phase portrait is drawn by choosing representative points in the plane, solving (43) numerically for the unique solution $K = k_2(x, y)$ and then iterating L_2. The phase portrait shows four saddle fixed points on the line $y = x$ (which is the so-called *symmetry line* [18] of the component involution (17)). An elliptic two-cycle is also visible.

Since $B_2(x, y, K)$ is a cubic polynomial in K, it is possible in this case to find $k_2(x, y)$ explicitly. Hence, if desired, L_2 can be written in a closed form that involves cube roots (highlighting the fact that a CDM map in general need not be a rational map).

Example 3.

$$B_3(x, y, K) = x^2 y^2 + (3K + 5)(x^2 + y^2) - 2(K + \cos K)xy + 1 = 0. \tag{45}$$

Part of the phase portrait of $L_3 : x' = y,\ y' = -x + \left.\frac{2(K+\cos(K))y}{y^2+3K+5}\right|_{K=k_3(x,y)}$ preserving this foliation is shown in figure 3. The biquadratic curves (45) are examples of the symmetric case of (36) with $\kappa(K) = \gamma(K)$ and $\mu(K) \equiv 1$. By construction, they satisfy $\gamma'(K) > 0$ and $\epsilon'(K)^2 < 4\gamma'(K)^2$ so that $\Delta(x, y, K)$ of (37) is negative except at $(0, 0)$ where $\frac{\partial B}{\partial K}(0, 0, K) = 0$ (i.e. (22) fails there). So in this example K is not defined at the origin but L_s clearly fixes the origin so we can extend the foliation and the mapping to include $(0, 0)$. Also note that (45), lacking the odd terms usually present in (16), possesses reflection symmetry in both the line $y = x$ *and* the line $y = -x$, as seen in figure 3.

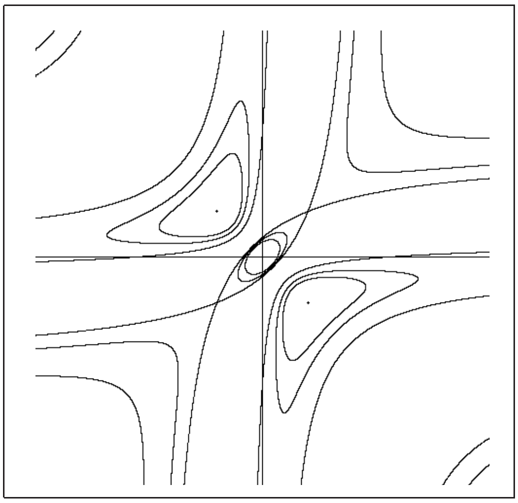

Figure 3. Phase portrait of L_3 of example 3 on the square $[-5, 5] \times [-5, 5]$.

Example 4.

$$\begin{aligned} B_4(x, y, K) = {} & \tfrac{1}{2}(K + \cos(K)\sin(K))\,x^2y^2 - \cos^2(K)(x^2y + xy^2) \\ & +\tfrac{1}{2}(K - \cos(K)\sin(K))(x^2 + y^2) + (K + 2\cos(K) \\ & +2K\sin(K) - \cos(K)\sin(K))\,xy + 2(\sin(K) \\ & -K\cos(K))(x + y) + \tfrac{1}{3}K(K^2 + 3) = 0. \end{aligned} \tag{46}$$

This example is created by integrating (40) with $r(K) = \cos K$, $s(K) = u(K) = \sin K$, $v(K) = K$ and $w(K) = 1$. One finds that irrespective of x and y, the dominant term of $B_4(x, y, K)$ at $\pm\infty$ is $\frac{1}{3}K^3$ so clearly $\lim_{K\to\pm\infty} B_4(x, y, K) = \pm\infty$.

Example 5.

$$\begin{aligned} B_5(x, y, K) = {} & \tfrac{1}{15}K(3K^4 + 10K^2 + 15)\,x^2y^2 + 2\mathrm{e}^K(3 + K^2 - 2K)(x^2y + xy^2) \\ & +\tfrac{1}{2}\mathrm{e}^{2K}(x^2 + y^2) + (\mathrm{e}^{2K} + \tfrac{1}{2}K^4 + K^2)\,xy \\ & +2\mathrm{e}^K(K - 1)(x + y) + \tfrac{1}{3}K(K^2 + 3) = 0. \end{aligned} \tag{47}$$

This example is created by integrating (40) with $r(K) = K^2 + 1$, $s(K) = u(K) = \mathrm{e}^K$, $v(K) = K$ and $w(K) = 1$. Again, irrespective of x and y, $\lim_{K\to\pm\infty} B_5(x, y, K) = \pm\infty$ so that $K(x, y)$ is well defined implicitly by $B_5(x, y, K) = 0$ for all points.

A nice feature of the condition (22) that allows further generation of examples is its additive nature. Thus suppose that we have $\{B_i(x, y, K) : 1 \leqslant i \leqslant N\}$ such that $\frac{\partial B_i}{\partial K}(x, y, K) \geqslant 0$, with at least one derivative when $i = j$ strictly >0. Then $B(x, y, K) = \sum_i c_i B_i(x, y, K)$, with c_i arbitrary non-negative numbers and $c_j > 0$, satisfies (22) and $B(x, y, K) = 0$ produces a foliation of the plane (modulo the endpoint conditions mentioned previously if B_i are not all polynomial). This again shows how the biquadratic foliations and their corresponding integrable maps can be made to depend on arbitrarily many parameters.

We end this section with one example illustrating the additive nature of (22).

Example 6.

$$\begin{aligned} B_6(x, y, K) = {} & (2 + 4K)x^2y^2 + K(x^2y + xy^2) + (4K + 5)\,(x^2 + y^2) \\ & -(K + 2\cos K)\,xy + K(x + y) + K + 1 = 0. \end{aligned} \tag{48}$$

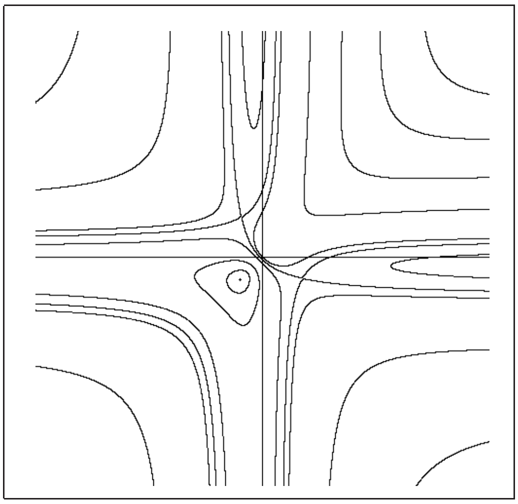

Figure 4. Phase portrait of L_6 of example 6 on the square $[-5, 5] \times [-5, 5]$.

Part of the phase portrait of $L_6 : x' = y,\ y' = -x + \frac{Ky^2-(K+2\cos(K))y+K}{(2+4K)y^2+Ky+4K+5}\Big|_{K=k_6(x,y)}$ preserving this foliation is shown in figure 4. It shows one elliptic point on the line $y = x$ in the third quadrant and a hyperbolic symmetric two-cycle in the second and fourth quadrants (where the curves are seen to intersect). Note that $B_6(x, y, K) = B_3(x, y, K) + B_{\mathrm{QRT}}(x, y, K)$ with $B_3(x, y, K)$ of (45) and $B_{\mathrm{QRT}}(x, y, K) = x^2y^2 + K(4x^2y^2 + (x^2y + xy^2) + (x^2 + y^2) + xy + (x + y) + 1)$ taken from a symmetric QRT foliation (32). We have $\frac{\partial B_3}{\partial K}(x, y, K) \geqslant 0$ (with equality at the origin) and $\frac{\partial B_{\mathrm{QRT}}}{\partial K}(x, y, K) > 0$ so that $\frac{\partial B_6}{\partial K}(x, y, K) > 0$.

4. Normal form for symmetric biquadratics

In this section we consider a normal form for the symmetric biquadratic foliation (16) and the associated mapping L_{s} of (24).

Consider the problem of looking for a zero x of a quartic polynomial

$$R(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0 \tag{49}$$

where a_i are constants, $a_4 \neq 0$. In the theory of elliptic integrals, there is a method due to Legendre that converts this problem into looking for a zero $\bar{x}$ of a quartic polynomial containing only even terms [6, 7]. That is, in the generic situation there exist real p and q such that if we set

$$\bar{x} = f_1(x) = \frac{p - x}{x - q} \tag{50}$$

then if x satisfies (49), $\bar{x}$ satisfies

$$\bar{R}(\bar{x}) = \bar{a}_4\bar{x}^4 + \bar{a}_2\bar{x}^2 + \bar{a}_0 = 0. \tag{51}$$

In (50), p and q are (real) functions of the (real and/or complex) zeros of $R(x)$. If a certain relationship between the zeros holds, (50) cannot be used but instead a translation

$$\bar{x} = f_2(x) = x + r \tag{52}$$

with r real can be used instead to give (51) (see [7, article 2] for details). Note that (50) and (52) are invertible transformations.

The above facts can be exploited in the case of the symmetric biquadratic foliation (16) to give, for each curve of this foliation, a transformation of this curve into a simpler *canonical* form. Furthermore, the *canonical* mapping that preserves this transformed curve is a simpler form of L_s of (24) (equivalently of M_s of (18) since L_s restricted to each curve is a McMillan map). The proof of the following result is given in [8] and exploits the fact that $B_s(x, x, K) = 0$ takes the form (49).

Proposition 2. *Suppose (x, y) satisfies $B_s(x, y, K) = 0$ with $B_s(x, y, K)$ the symmetric biquadratic (16). Then for each fixed K satisfying $\alpha(K) \neq 0$ there exists an invertible transformation $T_K(x, y) : (x, y) \mapsto (\bar{x}, \bar{y}) = (f_K(x), f_K(y))$ with f_K of the form (50) or (52) such that*

$$B_s^{\mathrm{can}}(\bar{x}, \bar{y}, K) = \bar{x}^2\bar{y}^2 + \bar{\gamma}(K)(\bar{x}^2 + \bar{y}^2) + \bar{\epsilon}(K)\bar{x}\bar{y} + \bar{\mu}(K) = 0. \tag{53}$$

Furthermore, if $L_s : (x, y) \mapsto (x', y')$ is the map (24) that preserves the curve $B_s(x, y, K) = 0$, then $L_s^{\mathrm{can}} = T_K \circ L_s \circ T_K^{-1} : (\bar{x}, \bar{y}) \mapsto (\bar{x}', \bar{y}')$ preserves the curve $B_s^{\mathrm{can}}(\bar{x}, \bar{y}, K) = 0$ and is given by:

$$L_s^{\mathrm{can}} : \quad \bar{x}' = \bar{y} \qquad \bar{y}' = -\bar{x} - \frac{\bar{\epsilon}(K)\bar{y}}{\bar{y}^2 + \bar{\gamma}(K)}. \tag{54}$$

With reference to this proposition, we note:

(i) if $\bar{\mu}(K) \neq 0$, a rescaling

$$(\bar{x}, \bar{y}) = |\bar{\mu}(K)|^{1/4} \ (\hat{x}, \hat{y}) \tag{55}$$

can be used so that $\hat{B}_s^{\mathrm{can}}(\hat{x}, \hat{y}, K) = 0$ takes the form (53) with $\hat{}$ variables and $\hat{\mu}(K) = \pm 1$ according as $\bar{\mu}(K) \gtrless 0$. Note that whenever $\bar{\mu}(K) = \pm 1$ in (53), L_s^{can} of (54) can be written so as to depend on only one of $\bar{\epsilon}(K)$ and $\bar{\gamma}(K)$ by using (53) to eliminate one in favour of the other;

(ii) the curve $B_s^{\mathrm{can}}(\bar{x}, \bar{y}, K) = 0$ has a double symmetry, being invariant under the interchange of $\bar{x}$ and $\bar{y}$ together with rotation by π (so that L_s^{can} commutes with $-Id$);

(iii) when $\bar{\gamma}(K) > 0\,(< 0)$, $B_s^{\mathrm{can}}(\bar{x}, \bar{y}, K) = 0$ is a bounded (unbounded) curve in the plane.

It is important to emphasize that proposition 2 allows one to simplify any given curve of the foliation $B_s(x, y, K) = 0$ with $\alpha(K) \neq 0$ to $B_s^{\mathrm{can}}(\bar{x}, \bar{y}, K) = 0$ and, correspondingly, to simplify the dynamics on it from being generated by L_s to L_s^{can}. It does *not* in general allow one to do this simultaneously on different curves and so infer L_s^{can} in a global way. In other words, although considered globally L_s is a CDM map, L_s^{can} is not. This is because the curves of $B_s^{\mathrm{can}}(\bar{x}, \bar{y}, K) = 0$ do not in general give a foliation; for different K, curves from this set intersect as can easily be checked on a few examples. This is because the global transformation of the plane built up by applying $T_K(x, y)$ on each curve labelled by K in the original foliation is not injective in general and does not map the original foliation to a new one. A sufficient condition for (53) to define a new foliation, and so for L_s^{can} to be a CDM map, is (39) (with bars added).

A special case when proposition 2 can be used in a global sense occurs when $T_K(x, y)$ is *independent* of K (i.e. $p(K)$ and $q(K)$ of (50), or $r(K)$ of (52), are independent of K). For example, the symmetric biquadratic foliation

$$x^2y^2 - 4(x^2y + xy^2) + \tfrac{31}{4}(x^2 + y^2) + 14xy - 27(x + y) + K = 0 \tag{56}$$

is preserved (cf (19) and (21)) by the symmetric McMillan mapping (see figure 5(*a*))

$$x' = y \qquad y' = -x - \frac{-4y^2 + 14y - 27}{y^2 - 4y + 31/4}. \tag{57}$$

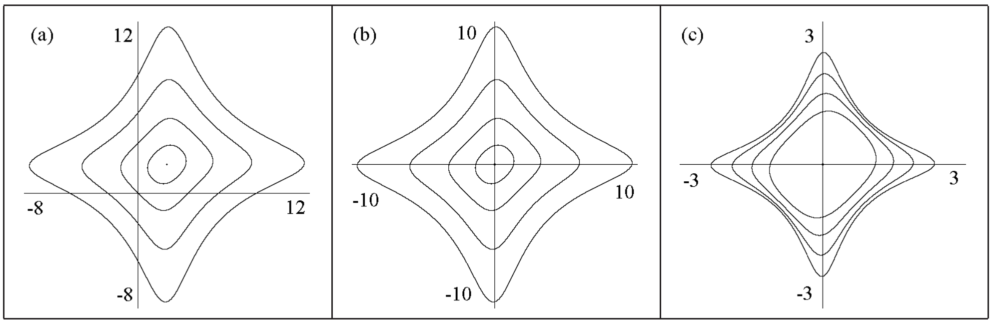

Figure 5. Four representative biquadratic curves invariant under (57) are shown in (*a*) together with their transformed versions in (*b*) and (*c*) which are preserved by, respectively, (59) and (64).

Using the transformation $T : (x, y) \mapsto (x-2, y-2)$ in (56) produces the transformed foliation B_s^{can} (bars omitted)

$$x^2y^2 + \tfrac{15}{4}(x^2 + y^2) - 2xy - 38 + K = 0 \tag{58}$$

with L_s^{can} another McMillan map given by (see figure 5(*b*))

$$x' = y \qquad y' = -x + \frac{2y}{y^2 + 15/4}. \tag{59}$$

As figure 5(*a*), (*b*) indicates, the centre of symmetry of the original foliation (56) is shifted to the origin in (58) so as to produce the second symmetry with respect to the origin.

In (58), a further global transformation can be made that is, interestingly, K dependent. It is easy to check that the range of K needed for (58) to foliate the plane is $K \in (-\infty, 38]$ ($K(0, 0) = 38$). So $\mu(K) = K - 38 \leqslant 0$ in (58) and we introduce, following (55), the scaling $(x, y) \mapsto |\mu(K)|^{-1/4}(x, y) = (38 - K)^{-1/4}(x, y)$ for $K < 38$ which now gives

$$x^2y^2 + \frac{15}{4\sqrt{38 - K}}(x^2 + y^2) - \frac{2}{\sqrt{38 - K}}xy - 1 = 0. \tag{60}$$

Under the scaling, (59) becomes the corresponding CDM mapping

$$x' = y \qquad y' = -x + \left.\frac{\frac{2}{\sqrt{38-K}}y}{y^2 + \frac{15}{4\sqrt{38-K}}}\right|_{K=k(x,y)}. \tag{61}$$

Despite the scaling varying on each curve of (58), the transformed family of curves (60) together with the origin is still a foliation of the plane. In fact, with reparametrizing

$$\tilde{K} := \frac{1}{\sqrt{38 - K}}, \tag{62}$$

equation (60) becomes

$$x^2y^2 + \tfrac{15}{4}\tilde{K}(x^2 + y^2) - 2\tilde{K}xy - 1 = 0 \tag{63}$$

which is linear in $\tilde{K}$. This is of the form of the QRT foliation (2) and, indeed, solving (60) for $\sqrt{38 - K}$ and substituting in (61) produces the symmetric QRT mapping (4) as follows:

$$x' = y \qquad y' = \frac{-8y + 15x(y^4 + 1)}{-15(y^4 + 1) + 8xy^3}. \tag{64}$$

The phase portrait of (64) in figure 5(*c*) shows that the curves of figure 5(*b*) are deformed into a new foliation.

This example of the application of a global K-dependent scaling to the foliation (58) can be generalized. Suppose we have a similar McMillan-type foliation

$$x^2y^2 + \gamma_0(x^2 + y^2) + \epsilon_0 xy + \mu_0 + K = 0 \tag{65}$$

with constants $\gamma_0 > 0$ and ϵ_0 satisfying $|\epsilon_0| < 2\gamma_0$. Then the range of K needed to foliate the plane is $K \in (-\infty, -\mu_0]$ with the maximum value occurring at the origin since $K(0, 0) = -\mu_0$. The scaling $(x, y) \mapsto |\mu(K)|^{-1/4}(x, y) = (-\mu_0 - K)^{-1/4}(x, y)$ for $K < -\mu_0$ can always be applied to (65) to transform it into a QRT foliation

$$x^2y^2 + \gamma_0\tilde{K}\,(x^2 + y^2) + \epsilon_0\tilde{K}\,xy - 1 = 0 \tag{66}$$

with $\tilde{K} = 1/\sqrt{-\mu_0 - K}$. Due to the condition $|\epsilon_0| < 2\gamma_0$, the origin is the only fixed point of the associated QRT map and is elliptic.

Finally, another application of proposition 2 in a global sense is to the symmetric QRT map itself. Consider the symmetric QRT foliation (32), that is, with A_0 and A_1 symmetric. Since $X \cdot A_0Y$ is a constant biquadratic independent of K, it follows that if $\alpha_0 \neq 0$ we can bring it to the form (53) with $\bar{\gamma}_0$, $\bar{\epsilon}_0$ constant and (from (55)) $\bar{\mu}_0 \in \{0, +1, -1\}$. In other words, applying an invertible global transformation $T : (x, y) \mapsto (\bar{x}, \bar{y})$ to (32) with $\alpha_0 \neq 0$ brings it to the form

$$\bar{B}_{\mathrm{QRT}}(\bar{x}, \bar{y}, K) = \bar{X} \cdot \bar{A}_0\bar{Y} + K(\bar{X} \cdot \bar{A}_1\bar{Y}) = 0 \tag{67}$$

in which $\bar{X} \cdot \bar{A}_0\bar{Y}$ depends on just two parameters. Consequently, the symmetric QRT foliation (2) and corresponding map (4) with $\alpha_0 \neq 0$ can always be made to depend on eight parameters instead of the notional 12 parameters present in (32).

5. Lax Pair

A symmetric mapping of the plane

$$x' = y \qquad y' = F(x, y) \tag{68}$$

with the identification $(x, y) \to (x_{n-1}, x_n)$ and $(x', y') \to (x_n, x_{n+1})$, $n \in \mathbb{Z}$ being discrete time, can be written as a second-order difference equation

$$x_{n+1} = F(x_{n-1}, x_n). \tag{69}$$

Recall that a Lax pair for a second-order difference equation (69) which is measure preserving is a pair of matrices $L_n(x_n, x_{n+1})$ and $M_n(x_n, x_{n+1})$ indexed by n satisfying

$$L_nM_{n-1} = M_{n-1}L_{n-1} \tag{70}$$

and such that the compatibility condition for the two sides of (70) to be equal yields the mapping (69). Since (70) gives the similarity of L_n and L_{n-1} at each step of the evolution, it follows that L_n and L_{n-1} share the same matrix polynomial invariants and that the coefficients of the characteristic polynomial:

$$P(\lambda) = \det(\lambda\mathbb{1} - L_n) \tag{71}$$

are invariant under the evolution (λ is the spectral parameter). If the coefficients are not all constant, this yields a non-trivial integral $I(x_n, x_{n+1})$ for (69).

Consider the second-order difference equation

$$x_{n+1} + x_{n-1} = -\frac{\beta(K)x_n^2 + \epsilon(K)x_n + \xi(K)}{x_n^2 + \beta(K)x_n + \gamma(K)} \tag{72}$$

which is the difference equation corresponding to M_{s} of (18) with $\alpha(K) \neq 0$ and *normalized* coefficient functions so that $\beta(K)$ in (72) corresponds to $\beta(K)$ in (18) *divided* by $\alpha(K)$ etc.

This difference equation includes the symmetric normal form (54) (omitting the bars) as a special case. For each value of K, this is a McMillan map. In [12] (see also [5, section 1.4]) a Lax pair was given for a symmetric McMillan map of the plane corresponding to (72) with $\beta = \xi = 0$ and $\gamma < 0$. We have been able to extend this to a Lax pair for (72) for each K.

Consider the matrices:

$$L_n(h) := \begin{pmatrix} a-e & b-c+x_{n+1} & 1 & 0 \\ 0 & 0 & b+c-x_n & 1 \\ h & 0 & a+e & b+c-x_{n+1} \\ h(b-c+x_n) & h & 0 & 0 \end{pmatrix} \tag{73}$$

and

$$M_{n-1}(h) := \begin{pmatrix} \frac{a-e}{b-c+x_n} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{a+e}{b+c-x_n} & 1 \\ h & 0 & 0 & 0 \end{pmatrix}. \tag{74}$$

Here a, b, c, e and h are arbitrary parameters (h is distinguished as a Floquet-type parameter following [5]). One finds that $L_n(h)M_{n-1}(h) = M_{n-1}(h)L_{n-1}(h)$ if and only if (72) holds with

$$\begin{aligned} &\beta(K) = -2c \qquad \gamma(K) = c^2 - b^2 \\ &\epsilon(K) = 4c^2 - 2a \qquad \xi(K) = 2(ac - eb + cb^2 - c^3). \end{aligned} \tag{75}$$

Solving (75) for a, b, c and e gives

$$\begin{aligned} &c = -\frac{\beta(K)}{2} \qquad a = \frac{\beta^2(K) - \epsilon(K)}{2} \qquad b = \frac{\sqrt{\beta^2(K) - 4\gamma(K)}}{2} \\ &e = \frac{\beta(K)\,(\epsilon(K) - \beta^2(K) + 2\gamma(K)) - 2\xi(K)}{2\sqrt{\beta^2(K) - 4\gamma(K)}}. \end{aligned} \tag{76}$$

Furthermore, writing out $\det(L_n)$ we obtain the following constant of the motion:

$$\begin{aligned} I(x_n, x_{n+1}) = {} & x_{n+1}^2 x_n^2 + \beta(K)\,(x_n^2 x_{n+1} + x_n x_{n+1}^2) + \gamma(K)\,(x_{n+1}^2 + x_n^2) \\ & + \epsilon(K)\,x_{n+1}x_n + \xi(K)\,(x_{n+1} + x_n). \end{aligned} \tag{77}$$

In other words, the iterates of (72) lie on the biquadratic curve $I(x_n, x_{n+1}) = D$, where D is a constant determined by initial conditions.

It is evident from (76) that (73), (74) represent a Lax pair for (72) for arbitrary values of $\beta(K)$, $\epsilon(K)$ and $\xi(K)$ with $\gamma(K)$ constrained by

$$\gamma(K) < \frac{\beta^2(K)}{4} \tag{78}$$

in order to keep the entries in the Lax pair real. For fixed K, this means we have a Lax pair for a very general symmetric McMillan map (72) (e.g. certainly for arbitrary $\beta(K)$, $\epsilon(K)$ and $\xi(K)$ and, say, $\gamma(K) \leqslant 0$ which clearly satisfies (78)). It follows from the previous section that for each K the map (72) can be reduced to the normal form (54) which corresponds to setting $\beta(K) = \xi(K) = 0$. It is seen from (76) and (78) that (73), (74) with $c = e = 0$ furnishes a Lax pair for the normal form (54) but only when $\gamma(K)$ is negative. Of course, the normal form (54) cannot be used in general as a global normal form for the dynamics.

Nevertheless, let us consider (72) in its own right as an example of an integrable symmetric CDM mapping preserving the family of curves (16) with $\alpha(K)$ there set equal to 1 and the other coefficients normalized by division by $\alpha(K)$. That is, *assume* that (16) defines a foliation of the plane by, for example, satisfying (22). Then (73) and (74) can be used to provide a (global)

Lax pair for the symmetric CDM mapping (72) whenever (16) can be solved explicitly for $K = k(x, y)$.

In particular, consider the 12-parameter symmetric QRT mapping (4) with f_i given by (34) with A_i symmetric (i.e. $\delta_i = \beta_i$, $\lambda_i = \xi_i$ and $\kappa_i = \gamma_i$ $(i = 0, 1)$) and α_0 and α_1 not both zero. Written as a second-order difference equation, it is

$$x_{n+1} = \frac{f_1(x_n) - x_{n-1} f_2(x_n)}{f_2(x_n) - x_{n-1} f_3(x_n)}. \tag{79}$$

From the discussion at the end of section 2 and the form of L_s in (24), it follows that, provided we exclude the single value $K = -\alpha_0/\alpha_1$ if $\alpha_1 \neq 0$, this QRT map can also be written in the form (72) with

$$\begin{aligned} \beta(K) &= \frac{\beta_0 + \beta_1 K}{\alpha_0 + \alpha_1 K} \qquad & \epsilon(K) &= \frac{\epsilon_0 + \epsilon_1 K}{\alpha_0 + \alpha_1 K} \\ \gamma(K) &= \frac{\gamma_0 + \gamma_1 K}{\alpha_0 + \alpha_1 K} \qquad & \xi(K) &= \frac{\xi_0 + \xi_1 K}{\alpha_0 + \alpha_1 K} \end{aligned} \tag{80}$$

and $K = k(x_{n-1}, x_n)$ given by

$$k = -\frac{\alpha_0 x_{n-1}^2 x_n^2 + \beta_0(x_{n-1}^2 x_n + x_{n-1} x_n^2) + \gamma_0(x_{n-1}^2 + x_n^2) + \epsilon_0 x_{n-1} x_n + \xi_0(x_{n-1} + x_n) + \mu_0}{\alpha_1 x_{n-1}^2 x_n^2 + \beta_1(x_{n-1}^2 x_n + x_{n-1} x_n^2) + \gamma_1(x_{n-1}^2 + x_n^2) + \epsilon_1 x_{n-1} x_n + \xi_1(x_{n-1} + x_n) + \mu_1}. \tag{81}$$

Substituting (80) into (76) and whence into (73) and (74) furnishes a Lax pair for the symmetric QRT mapping, provided we use $K = k(x_n, x_{n+1})$ in (73) and $K = k(x_{n-1}, x_n)$ in (74), and observe the constraint

$$4(\alpha_0 + \alpha_1 K)(\gamma_0 + \gamma_1 K) < (\beta_0 + \beta_1 K)^2 \tag{82}$$

obtained from (78) together with (80). One finds that the relationship between the integral $I(x_{n-1}, x_n)$ of (77) obtained from the Lax pair and $k(x_{n-1}, x_n)$ is

$$I = -\frac{\mu_0 + \mu_1 k}{\alpha_0 + \alpha_1 k}. \tag{83}$$

Condition (82) places some restrictions on the subset $\{\alpha_0, \alpha_1, \beta_0, \beta_1, \gamma_0, \gamma_1\}$ of six parameters in (79). The condition can be also written in terms of the polynomial

$$P(K) = (\beta_1^2 - 4\alpha_1\gamma_1)K^2 + 2(\beta_0\beta_1 - 2(\alpha_0\gamma_1 + \alpha_1\gamma_0))\, K + (\beta_0^2 - 4\alpha_0\gamma_0) > 0 \tag{84}$$

where $P(K)$ has the discriminant

$$\Delta(K) = 16\, [(\alpha_0\gamma_1 - \alpha_1\gamma_0)^2 + (\beta_0\alpha_1 - \beta_1\alpha_0)(\beta_0\gamma_1 - \beta_1\gamma_0)]. \tag{85}$$

Condition (84) must hold for the range of k in (81) achieved over the (x_{n-1}, x_n) plane. This range can be worked out for a particular example, but let us consider the sufficient restrictions in terms of γ_0 and γ_1 that result from imposing (84) for all K. To make the leading coefficient of $P(K)$ non-negative we introduce the parameter $t \geqslant 0$ such that

$$\beta_1^2 - 4\alpha_1\gamma_1 = t^2. \tag{86}$$

If $\alpha_1 \neq 0$, this leads to the parametrized restrictions

$$\gamma_1(t) = \frac{\beta_1^2 - t^2}{4\alpha_1} \tag{87}$$

and

$$\gamma_0(t) \in (r_-(t), r_+(t)), \tag{88}$$

where

$$r_{\pm}(t) := \frac{2\alpha_1\beta_0\beta_1 - \alpha_0(\beta_1^2 + t^2) \pm 2\,t\,|\,\beta_0\alpha_1 - \beta_1\alpha_0\,|}{4\alpha_1^2}. \tag{89}$$

On the other hand, if $\alpha_1 = 0$ (and so $\alpha_0 \neq 0$ by assumption), we find that (84) holding for all K leads to (again $t \geqslant 0$)

$$\gamma_0(t) = \frac{\beta_0^2 - t^2}{4\alpha_0} \tag{90}$$

and

$$\gamma_1(t) \in (s_{\min}(t), s_{\max}(t)), \tag{91}$$

where $s_{\min}$ and $s_{\max}$ are the smaller and larger values of

$$s_{\pm}(t) := \frac{\beta_1(\beta_0 \pm t)}{2\alpha_0}. \tag{92}$$

Some special (limiting) cases of these parametrized restrictions can be obtained directly from (78) together with (80): (i) when $\beta_1 = \beta_0 = 0$ we find that γ_i and α_i $(i = 0, 1)$ are proportional by the same negative number so that $\gamma(K) < 0$; (ii) we can always take $\gamma_1 = \gamma_0 = 0$ in (87), (88) and (90), (91) for appropriate t so that $\gamma(K) = 0$.

The end result of the previous analysis is that we have a Lax pair for the symmetric QRT map (72) with (80), (81) and $\alpha(K) \neq 0$ that holds in general for ten of the 12 parameters being arbitrary and two parameters confined to an interval.

6. Concluding remarks

In this paper, we have introduced integrable mappings of the plane that preserve biquadratic foliations and are of McMillan type on each curve. We have concentrated on the symmetric foliation and corresponding symmetric map. This case is easier to deal with than the asymmetric case. In [8], more details on both the symmetric and asymmetric maps will be given, in particular the normal forms of the biquadratic curves and the possibility to parametrize the dynamics on them by Jacobian elliptic functions will be discussed in detail. We are also investigating a less restrictive condition than (22) for ensuring condition F.

An obvious extension of the ideas of this paper which is also being investigated is to higher-dimensional mappings [17]. In this respect, McMillan maps in $2P$ dimensions, $P \geqslant 2$, are given in [12]. Higher-dimensional analogues of the QRT planar maps (3) and (4) were considered in [3, 13]. What is obvious from the present paper is that many of the properties of QRT maps can be found by treating them as generalizations of McMillan maps.

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